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The method of fundamental solutions for solving free boundary saturated seepage problem ☆

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ABSTRACT

The problem of seepage flow through a dam is free boundary problem that is more conveniently solved by a meshless method than a mesh-based method such as finite element method (FEM) and finite difference method (FDM). This paper presents method of fundamental solutions, which is one kind of meshless methods, to solve a dam problem using the fundamental solution to the Laplace's equation. Solutions on free boundary are determined by iteration and cubic spline interpolation. The numerical solutions then are compared with the boundary element method (BEM), FDM and FEM to display the performance of present method.

1. Introduction

The two-dimensional steady state saturated isotropic seepage flow with free boundary is described by the Laplace equation necessary boundary conditions. Previous works, the methods are to solve the unconfined seepage problem; it can be classified as analytical and numerical methods. The analytical solution can be obtained by using the theory of analytical function for liner ordinary differential equations [1,2]. It is only valid for two-dimensional problem but it cannot be used in case of complex geometrics and three-dimensional problems.

Conventionally numerical methods used to solve such problem included FDM [3] and FEM [4–8]. These methods are all mesh-dependent methods because they require boundary-fitted mesh generation. Alternative numerical methods include BEM [9] and MFS [10,11]. Both methods do not require boundary-fitted mesh, which results in considerable simplification of the preprocessing step. MFS has additional advantages over BEM in that it requires only boundary node placement instead of boundary mesh generation, and it does not require evaluation of near singular integrals [12]. The basic idea of MFS is to approximate the solution by forming a linear combination of known fundamental solutions with sources located outside the problem domain.

Previously, Chantasiriwan [13] investigated numerically both one- and two-phase Stefan problem subject to specification of boundary temperature, heat flux or energy using MFS. The numerically obtained results showed good agreement with the available analytical solutions. Kolodziej et al. [11] implemented the MFS with radial basis functions to solve a heat source problem for arbitrary domains, the numerical results showed that the MFS is an accurate and reliable numerical technique for the solution of the inverse heat source problem.

In order to study seepage problem, accurately defining the position of free boundary is very important and necessary. In the past, many researchers utilized several methods to determine the location of free boundary such as Aitchison [3], and Westbrook [4] used FDM and FEM respectively, to solve the position of the free boundary, respectively. The conventional BEM was then used to study the seepage flow through the porous media by Liggett and Liu [14], and also BEM using hypersingular equations was proposed by Chen et al. [15].

In this paper, free boundary is regarded as a moving boundary with the over-specified boundary conditions, and MFS is used to find the location of free boundary. The numerical results of present method are also compared with FDM, FEM, and BEM solutions.

2. Mathematical formulations

The seepage problem of water flow through a saturated porous medium dam with tail water is shown in Fig. 1. The free boundary is defined as the boundary line or interface between the saturated-wet and dry soils. In order to reduce complexity of the phenomena to analyze flow field in the soil, several assumptions are introduced as following:

(1) Soil in the flow field is homogeneous and isotropic.
(2) Capillary and evaporation effects are neglected.
(3) Two dimensional steady-state flow.
(4) The flow follows Darcy's law.
(5) Hydraulic conductivity or permeability of the soil is constant isotropic seepage flow.

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Nomenclatures

n	direction cosine (–)
G	fundamental solutions (m)
p	pressure (N/m^2)
u	component of velocity in x direction (m/s)
v	component of velocity in y direction (m/s)
x, y	cartesian coordinates

Greek letters

γ	specific weight (N/m^2)
φ	velocity potential (m^2/s)
ψ	stream function (m^2/s)

Subscripts

i, j	index

Abbreviations

BEM	boundary element method
FDM	finite difference method
FEM	finite element method
MFS	method of fundamental solutions

The governing equation of two-dimensional steady-state isotropic seepage in terms of the velocity potential and the streamline function can be described by the Laplace's equation as

$$\nabla^2\varphi = 0 \tag{1}$$

$$\nabla^2\psi = 0. \tag{2}$$

It is found that they are orthogonal to each other [16]. The component u-velocity and v-velocity in x- and y-direction, respectively, can be expressed as

$$u = -\frac{\partial\varphi}{\partial x}, \quad v = -\frac{\partial\varphi}{\partial y}. \tag{3}$$

The velocity potential function or piezometric head can be written as

$$\varphi = y + \frac{p}{\gamma} \tag{4}$$

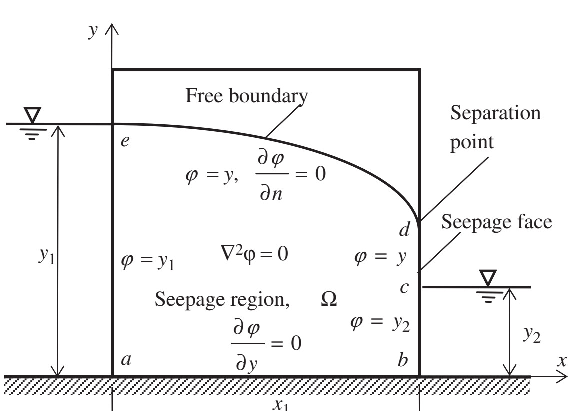

Fig. 1. Flow through a 2D rectangular dam.

where y is the position, p is the pressure, and $\gamma$is the specific gravity of fluid [16]. Therefore, the boundary conditions are presented in Fig. 1 as

$$\varphi = y_1 \quad \text{on } a-e \tag{5}$$

$$\varphi = y_2 \quad \text{on } b-c \tag{6}$$

and the remaining free boundary conditions

$$\varphi = y \quad \text{on } c-d \text{ and } d-e \tag{7}$$

$$\frac{\partial\varphi}{\partial n} = 0 \text{ on } d-e. \tag{8}$$

Boundary conditions at the free boundary are over-specified. In the following section, this boundary will be determined by using MFS with the initial guess of free boundary.

3. Numerical methods

3.1. The method of fundamental solutions

For basic idea of MFS is to express φ as linear combination of fundamental solutions [10]. Consider Fig. 1, let Ω is seepage region that be a bounded, simply connected domain in R^2 with boundary Γ. On boundary b–c, c–d, and a–e are Dirichet boundary, and boundary a–b is Neumann boundary. Boundary d–e is combined Dirichet and Neumann boundary, or is called as Robin boundary. For these boundaries can generally expressed as

$$\varphi = f(x,y) \quad \text{for } (x,\ y) \text{ on } \Gamma_1 \tag{9}$$

$$n_x\frac{\partial\varphi}{\partial x} + n_y\frac{\partial\varphi}{\partial y} = (x,y) \quad \text{for } (x,\ y) \text{ on } \Gamma_2 \tag{10}$$

where direction cosine n_x and n_y are x-, and y-components, respectively, of the outward normal unit vector. The fundamental solution satisfies the solution of Laplace's equation as

$$G\left(P_i, Q_j\right) = \frac{1}{2\pi}\, log\ r_{ij} \quad \left(P_i{\in}\Omega,\ ,Q_j,{\in}\hat{S}\right) \tag{11}$$

where

$$r_{ij} = \sqrt{\left(x_i - \xi_j\right)^2 + \left(y_i - \eta_j\right)^2} \tag{12}$$

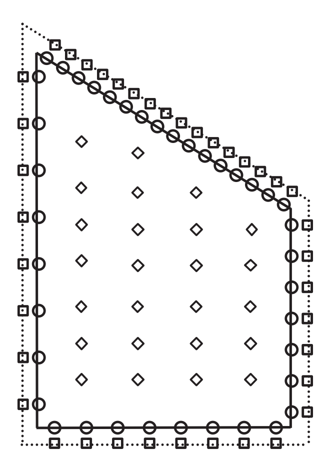

Fig. 2. The distributions of collocation points (white circles), source points (white squares), and internal domain points (white diamonds).

is a Euclidian distance between collocation point and source point, and (ξ_j, η_j) are coordinates of source points that located outside the domain shown in Fig. 2.

Since seepage problem included free boundary must be solved iteratively. Suppose that after the n^{th} iteration, value of $\varphi_i^{(n)}$ are known, values of $\varphi_i^{(n+1)}$ at $(n+1)^{\text{th}}$ iteration are to be determined. Therefore, the approximate solution of Eq. (1) can be represented by a linear combination of fundamental solution as

$$\varphi_i^{(n+1)} = \sum_{j=1}^{N} a_j^{(n+1)} G\left(P_i, Q_j\right) \quad Q_j \in \hat{S} \tag{13}$$

where N be number of nodes in boundary domain. Substituting Eq. (13) into Eqs. (9) and (10) results in a system of equations:

$$\sum_{j=1}^{N} a_j^{(n+1)} G\left(P_i, Q_j\right) = f(x_i, y_i) \qquad (i = 1, 2, \ldots, N_1) \tag{14}$$

$$\sum_{j=1}^{N} a_j^{(n+1)} \left[n_x \frac{\partial}{\partial x} G\left(P_i, Q_j\right) + n_y \frac{\partial}{\partial y} G\left(P_i, Q_j\right) \right] = g(x_i, y_i) \tag{15}$$

where N_1 and N_2 are the number of nodes on boundary $\Gamma_1$and Γ_2, respectively, and $N = N_1 + N_2$. Hence, $a_j^{(n+1)}$ can be determined. Direction cosine n_x and n_y in Eq. (10) or Eq. (15) on free boundary can be expressed as

$$n_x = \cos\alpha \tag{16}$$

$$n_y = \cos\beta \tag{17}$$

respectively, and further details shown in Fig. 3.

Therefore, each of iteration, direction cosines of free boundary nodes are to be determined by using boundary nodes and central boundary nodes as displayed in Fig. 3. For central boundary nodes are interpolated by cubic spline interpolation (CBI) [17,18], when x-coordinate of those nodes are specifically known. Free boundary is also obtained by this interpolation technique. CBI is chosen because it uses third degree polynomials to connect the data points which often results in strikingly smooth curve fitting. For separation point is shown in Fig. 1, it is calculated by second degree polynomials after free boundary obtained for each of iteration.

Since the free boundary has over specified boundary conditions, it will be determined iteratively by using initial guess for free boundary as shown in Fig. 4. Additionally, Fig. 4 shows positions of source points in the space coordinates. It can be seen that the number of source points is the number of boundary nodes (N). The N source points have the space coordinates as

$$(\xi_i, \eta_i) = (x_i, y_i) + BF \cdot \left(n_{x,i}, n_{y,i}\right) \tag{18}$$

where BF is body factor constant, for this paper, let BF is equal to 1.0 to determine coordinate of source points. Each of source points is also located on an imaginary boundary, which is larger than the actual boundary. The free boundary location is determined by checking the criterion of convergence as following

$$\varepsilon = \frac{\sqrt{\sum_{i=1}^{m} \left(\varphi_i^{n+1} - \varphi_i^{n}\right)^2}}{\sqrt{\sum_{i=1}^{m} \left(\varphi_i^{n}\right)^2}} \tag{19}$$

where the symbol m is the total number of boundary nodes on the moving surface, and the allowable tolerance used in this paper is 10^{-4} as same as Chen et al. [5]. The flowchart of iteration procedure is also displayed in Fig. 5.

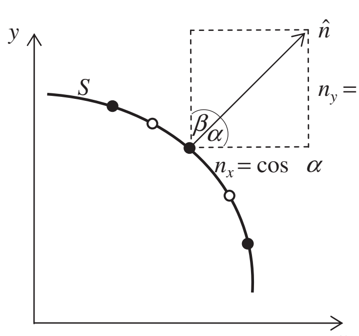

Fig. 3. Direction cosine, and locations of boundary nodes (black circles) and central boundary nodes (white circles).

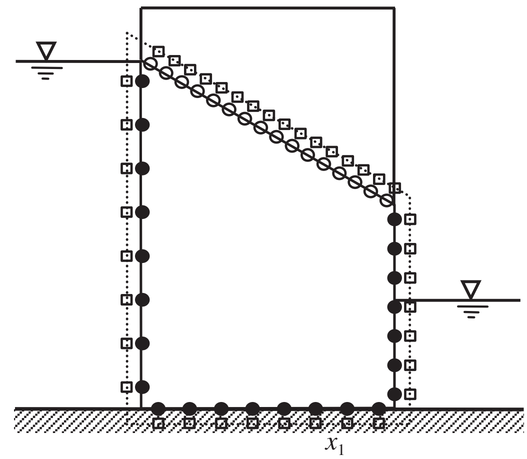

Fig. 4. Initial model: boundary nodes (black circles), initial free boundary nodes (white circles), and source points (white squares).

3.2. Finite element method

A standard finite element method is given in this section for two-dimensional seepage flow domain Ω. The weak form of Eq. (1) is obtained by multiply both sides of this equation by arbitrary continuous function $\overline{\varphi}$ and integrating over the domain Ω, applying the divergence theorem [5–8] as

$$\int_{\Omega} (\nabla\overline{\varphi})^T \nabla\varphi d\Omega = 0. \tag{20}$$

It is noted that the weak form is nonlinear since the flow domain Ω is unknown such as the location of free boundary and the separation point are also unknown, although Eq. (20) is apparently linear in φ, it will be determined iteratively by using initial guess for free boundary. In finite element method the dependent variable, the velocity potential φ, is approximated by

$$\varphi = \sum_{i=1}^{n} \varphi_i N_i \tag{21}$$

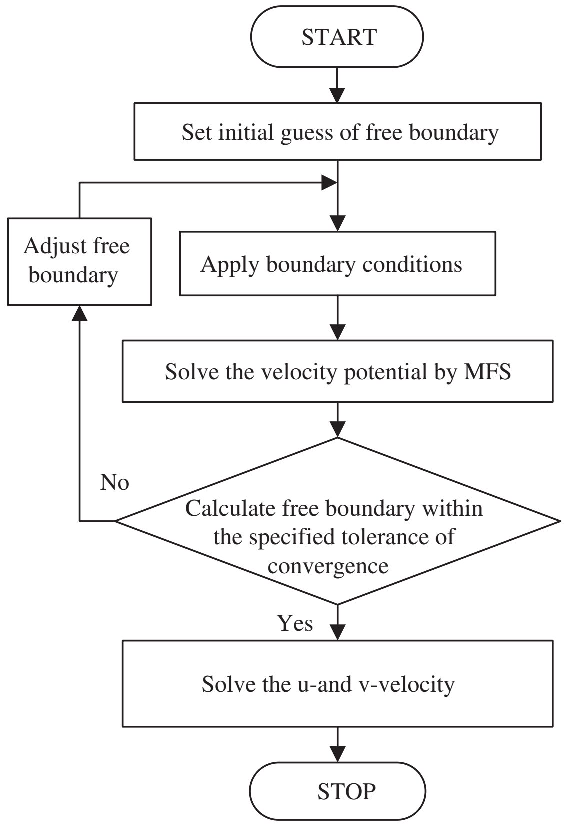

Fig. 5. Flow chart of iteration procedure.

where usually the φ_i are the nodal values of φ, N_i are appropriate shape functions (interpolation function) defined piecewise element by element, and n is the total number of nodes. The linear algebraic equation system is derived by Galerkin's method as

$$K_{ij}\varphi_j = 0 \quad (i,j = 1,2, \ldots,n) \tag{22}$$

where n is the number of nodes of finite element mesh and K_{ij} is a global matrix coefficient given by

$$K_{ij} = \int_\Omega \frac{\partial N_i}{\partial x_k}\frac{\partial N_j}{\partial x_l}d\Omega \quad (i,j = 1,2, \ldots,n)(k,l = 1,2,3). \tag{23}$$

3.3. Boundary element method

The essence of a boundary element method implemented this problem is to transform the variables from area variables to boundary ones. The simplest approach is to use Green's second identity. Here one can also introduce the idea of multiplying Eq. (1) by a fundamental solution of Lapalce's equation G. Applying Green's second identity to φ and G results in the following transformation from an area integral $d\Omega$ to a line integral $d\Gamma$ [9,15] as

$$\int_\Omega \left(G\nabla^2\varphi - \varphi\nabla^2 G\right)d\Omega = \int_\Gamma \left(G\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial G}{\partial n}\right)d\Gamma \tag{24}$$

where n is the unit outward normal and $\partial/\partial n$ is the derivative in the direction of the outward normal. Using weighted residual technique, a residual R function is set to be a fundamental solution G, Then Eq. (24) is obtained as

$$\varphi = \int_\Gamma G\frac{\partial\varphi}{\partial n}d\Gamma - \int_\Gamma \varphi\frac{\partial G}{\partial n}d\Gamma. \tag{25}$$

Discretizing the boundary Γ into N_e element, it is obtained as

$$\varphi_i = \sum_{k=1}^{N_e} I_k \tag{26}$$

$$I_k = \int_{\Gamma_k}\left[\frac{\partial\varphi}{\partial n}G(s,x_i,y_i) - \varphi(s)\frac{\partial G}{\partial n}(s,x_i,y_i)\right]J(s)ds \tag{27}$$

where Γ_k is a boundary element at k and N_e is a total boundary element. The velocity potential function φ can be approximated by interpolation function N_l as following:

$$\varphi(s) = \sum_{i=1}^{m} N_l(s)\varphi_{k,l} \tag{28}$$

where it is should be $m=3$ for quadratic element.

4. Analytical method

In this case, the analytical solution of the free boundary can be given in following form [1,2]

$$x = x_1 - \int_0^\chi \frac{\zeta\left(\sin^2\chi\right)\sin\chi d\chi}{\sqrt{(1-\alpha\sin^2\chi)(1-\beta\sin^2\chi)}} \tag{29}$$

$$y = y_1 + y_2 + \int_0^\chi \frac{\zeta\left(\cos^2\chi\right)\sin\chi d\chi}{\sqrt{(1-\alpha\sin^2\chi)(1-\beta\sin^2\chi)}} \tag{30}$$

$$0 \le \chi \le \pi/2 \tag{31}$$

where $\zeta(\chi)$ is the complete elliptic integral of the first kind; $\alpha,\beta \in (0,1)$ are parameters that define problem; the domain parameter of y_1, y_2 and x_1 are defined as

$$y_1 = \int_0^{\pi/2}\frac{\zeta\left(\alpha + (\beta-\alpha)\sin^2\chi\right)d\chi}{\sqrt{\beta-\alpha + (1-\beta)\sin^2\chi}} \tag{32}$$

$$y_2 = \sqrt{\alpha}\int_0^{\pi/2}\frac{\zeta\left(\alpha\sin^2\chi\right)\sin\chi d\chi}{\sqrt{(1-\alpha\sin^2\chi)(\beta-\alpha\sin^2\chi)}} \tag{33}$$

$$x_1 = \int_0^{\pi/2}\frac{\zeta\left(\alpha + (1-\beta)\sin^2\chi\right)d\chi}{\sqrt{1-\alpha-(\beta-\alpha)\sin^2\chi}}. \tag{34}$$

The length of the seepage surface is obtained as

$$y_0 = \int_0^{\pi/2}\frac{\zeta\left(\cos^2\chi\right)\sin\chi\cos\chi d\chi}{\sqrt{(1-(1-\alpha)\sin^2\chi)\cdot(1-(1-\beta)\sin^2\chi)}} \tag{35}$$

5. Results and discussion

In the following, the proposed numerical technique MFS is used to solve a two-dimensional unconfined seepage flow problems of rectangular dam with tail water.

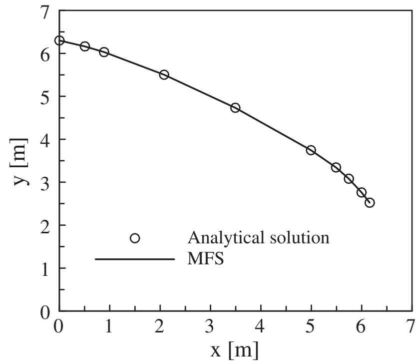

Fig. 6. Comparison of free boundary from analytical and MFS solution.

5.1. Numerical validation

In order to verify effectiveness of MFS, there two test problems of rectangular dam with tail water that are considered. The MFS results should be compared with analytical solution in first test, and conventional methods as FDM, FEM and BEM solutions in second test.

5.1.1. First test problem

In analytical method, after taking $\alpha = 0.3$, $\beta = 0.9$ and performing numerical integration, the physical parameters of the problem: $y_1 = 6.3014\,m$, $y_2 = 1.2359\,m$, $x_1 = 6.1592\,m$ and the length of seepage face $y_0 = 1.2868\,m$. The analytical results of the coordinates of the free boundary nodes are calculated by Eqs. (29) and (30). Fig. 6 shows the present simulated MFS results. The solutions obtained agree closely with the analytical results.

5.1.2. Second test problem

Consider seepage problem where the upper hydraulic head $y_1 = 24$ m, the lower hydraulic head $y_2 = 4$ m, and the width of dam $x_1 = 16$ m. There are 70 nodes uniformly distributed in the initial guess domain with grid spacing of 1.0 and separation point is assumed at $y = 14$ m. The present numerical solutions of free boundary are then compared with those of Aitchison [3], Westbrook [4], and Chen et al. [15] as shown Table 1. The number of iterations of present method is obtained by 22. It can be seen that that MFS is capable to calculate free boundary agree with other methods.

The separation point at $x = 16.0\,m$ is interesting and important since a singular point due to the intersection of the free boundary and seepage surface. In addition, this point presents an important role in term of dam stability. It is predicted by MFS and compared with other methods as shown Table 2.

Table 1
Free boundary obtained by different methods.

x (m)	MFS	FDM [3]	FEM [4]	BEM [15]
1	23.75	23.74	23.64	23.74
2	23.41	23.41	23.32	23.40
3	23.03	23.02	23.06	23.01
4	22.59	22.59	22.52	22.52
5	22.12	22.12	22.12	22.09
6	21.60	21.60	21.55	21.57
7	21.04	21.04	21.07	21.00
8	20.44	20.43	20.36	20.39
9	19.79	19.78	19.81	19.73
10	19.08	19.08	19.07	19.02
11	18.32	18.31	18.26	18.24
12	17.50	17.48	17.45	17.39
13	16.59	16.57	16.45	16.45
14	15.58	15.54	15.51	15.39
15	14.40	14.39	14.33	14.09
16	12.88	12.79	Not shown	12.68

Table 2
The separation point calculated by different methods.

Reference	Height (m)
Present MFS	12.88
FDM [3]	12.79
FEM [4]	Not shown
BEM [15]	12.68

5.2. Flow analysis

As in two previous test problems, the rectangular dam is homogeneous and isotropic. Water flows through the dam from a reservoir on the upstream side to the downstream side as shown in Fig. 4. Fluid particles move along a streamline always in the same direction from the upper to the lower reservoir that is the direction of x increasing in Fig. 7. Thus the fluid velocity is positive in the flow region Ω and the velocity potential φ is decreased along any streamline as display in Fig. 7. The velocity field in Fig. 7(c) shows zero normal component on the free boundary. The velocity potential function contours in Fig. 7(d) are normal to the free boundary and the bottom boundary, which are stream lines.

6. Conclusion

In this paper, it is shown how to use MFS to solve the problem of two-dimensional steady-state isotropic seepage flow. A generalized mathematical model and an effective calculation procedure are proposed. For two test problems indicate the successful implementation of numerical procedure. The free boundary and separation point can be obtained. Although it is only considered solving dam problem, MFS can be applied to more general free boundary problems.

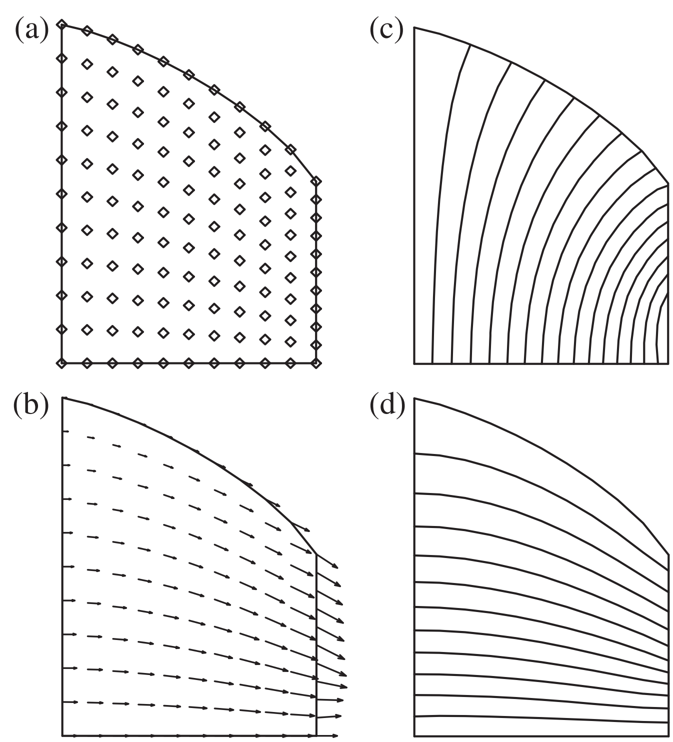

Fig. 7. Flow distribution: (a) converged shape and domain points (white diamonds); (b) velocity field in the converged shape; (c) the velocity potential distribution; and (d) the stream function distribution.

Acknowledgments

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